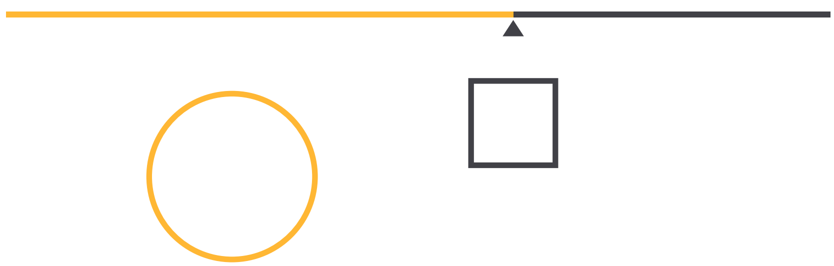
The Wire Problem - A Calculus Approach

The objective of this activity is to optimize the area formed when a wire of a given length is cut and used to create two shapes.

A wire that is 40 cm long will be used to form a circle, a square, or both. The wire may be cut so that one section is used to create a circle and the other section is used to create a square.



How should the wire be used to form shapes with a maximum combined area? How much of the wire should be used to create the circle? How much should be used to create the square?

There are many ways that students may approach this problem. Consider selecting and sequencing techniques when having students present their work. **In calculus, using the First Derivative Test is ideal.**

Overall, this problem consists of selecting where to cut the wire, constructing a square from one piece and a circle from the other, and computing the area of each shape.

Let x be the distance from the left end of the wire as shown below.

Cutting at the location indicated above will result in two pieces of wire, one with length x and the other with length $40 - x$. Let the piece of wire with length x be used to construct the circle and the piece of wire with length $40 - x$ be used to construct the square.

For the circle, the length of the wire x corresponds to the circumference of the circle. That is, $x = 2\pi r$. To determine the area of the circle, we must first solve for r

$$x = 2\pi r$$

$$\frac{x}{2\pi} = r$$

Now we find the area of the circle using $A = \pi r^2$

$$A_{circle} = \pi\left(\frac{x}{2\pi}\right)^2$$

$$A_{circle} = \frac{x^2}{4\pi}$$

For the square, the length of the wire $40 - x$ corresponds to its perimeter. That is, $P = 40 - x$. To calculate the area of the square, we must first determine the side length of the square. Since a square has 4 congruent sides, we have:

$$4s = 40 - x$$
$$s = \frac{40 - x}{4}$$

Now we find the area of the square using s^2

$$A_{square} = \left(\frac{40 - x}{4}\right)^2$$

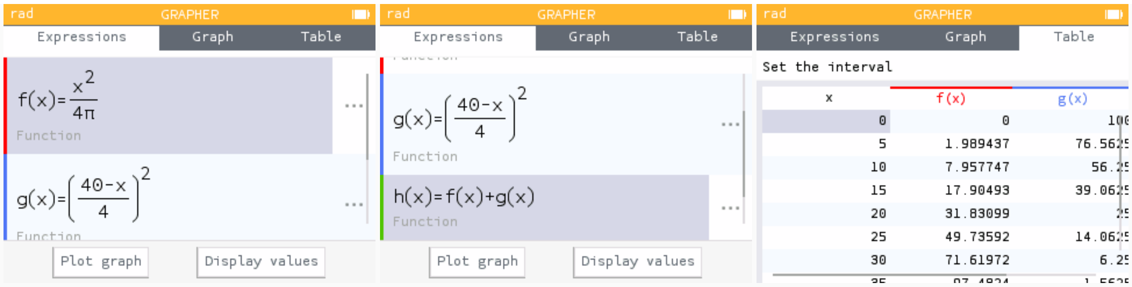
The total combined area can be found by adding the two areas

$$A = \frac{x^2}{4\pi} + \left(\frac{40 - x}{4}\right)^2$$

Creating a Table

Using the area formula above, students may consider a guess-and-check method by trying out different values x. A table of values for x and its corresponding areas can be constructed.

Using the Grapher app, input functions to compute the area of the circle, the area of the square, and the total area. Then view the Table tab and input possible values for x.

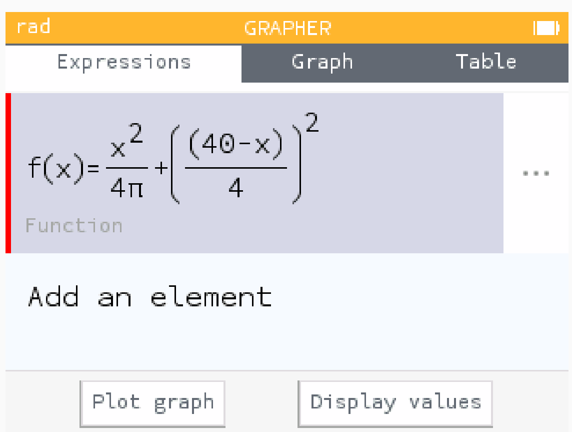


x	Area of Circle	Area of Square	Total Area
0	0	100	100
5	1.989	76.563	78.552
10	7.958	56.25	64.208
15	17.905	39.063	56.967
20	31.831	25	56.831
25	49.736	14.063	63.794
30	71.62	6.25	77.87
35	97.482	1.563	99.045
40	127.324	0	127.324

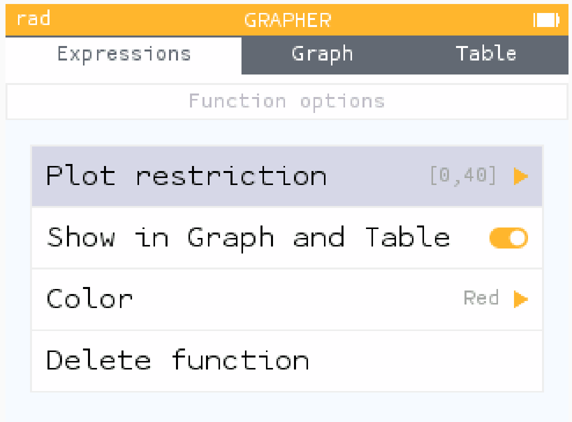
Using this approach, students will find the maximum at $x = 40$. That is, when the entire piece of wire is used to construct a circle, the area is maximized at 127.324cm^2.

Graphing

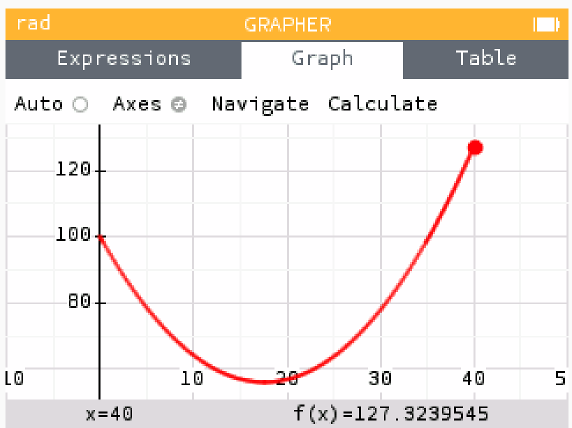
Using the area formula above, students may graph the function $y = \frac{x^2}{4\pi} + \left(\frac{40-x}{4}\right)^2$ and look for maximums. Using the Functions app, enter the function $f(x) = \frac{x^2}{4\pi} + \left(\frac{40-x}{4}\right)^2$.



Before viewing the Graph, adjust the plot range in the Function options menu to match the *practical domain* for this scenario.



From the graph of the function, we see the maximum occurs at the right endpoint $(40, 127.324)$.

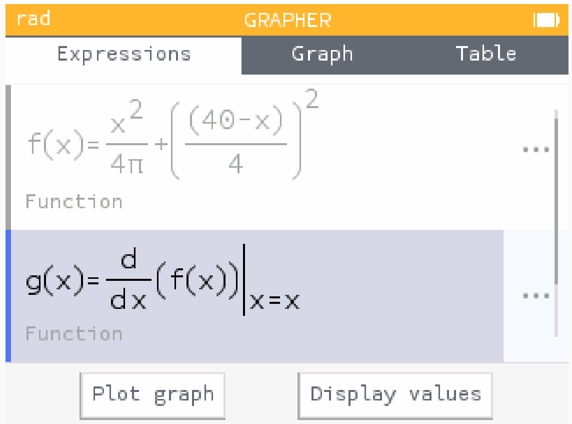


That is, when the entire piece of wire is used to construct a circle, the area is maximized at 127.324cm^2.

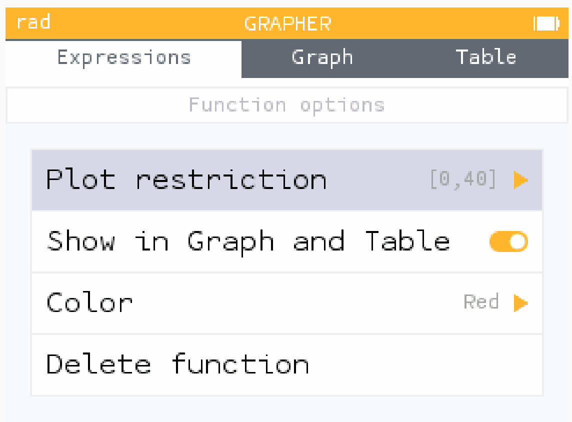
First Derivative Test

Using the area formula above, students may use the first derivative test to determine the extrema of the function $f(x) = \frac{x^2}{4\pi} + \left(\frac{40-x}{4}\right)^2$. The Extreme Value Theorem states that since $f(x)$ is continuous on $[0, 40]$, then $f(x)$ must have a maximum on the interval. This problem demonstrates the importance of checking the endpoints of the interval.

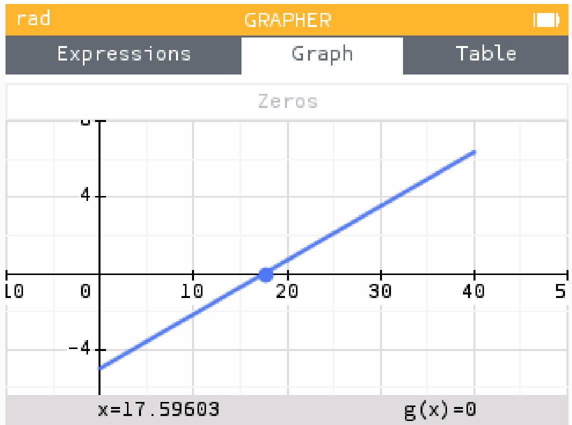
Using the Functions app, enter the function $f(x) = \frac{d}{dx}\left(\frac{x^2}{4\pi} + \left(\frac{40-x}{4}\right)^2\right)\Big|_{x=x}$.



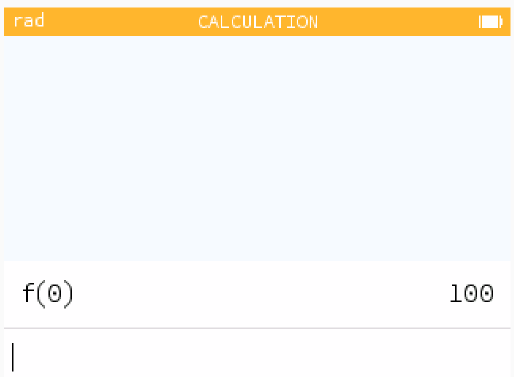
Before viewing the Graph, adjust the plot range in the Function options menu to match the *practical domain* for this scenario.

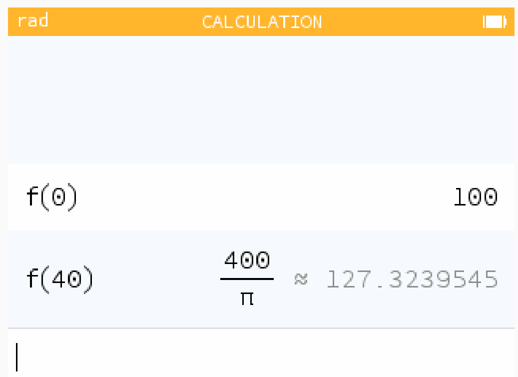


Graphing the first derivative, we see a zero occurs at $(17.596, 0)$. Therefore, our only critical point is at $x = 17.596$. However, since the derivative is *negative* before $x = 17.596$ and *positive* after $x = 17.596$, this critical point corresponds to a minimum.



At this point, we should check the endpoints. Using the Calculation app, we can store our endpoints as x and evaluate the function.





We see that the right endpoint outputs the largest value. That is, when the entire piece of wire is used to construct a circle, the area is maximized at 127.324cm^2.

Additional Prompts

The following alterations can be used depending on your learning objectives:

1. Change the **length** of the wire for each student or group of students.
2. Replace the length of the wire with a **parameter** L.
3. Instruct students to find the **minimum** combined area instead of the maximum area.
4. Use the wire to form a **circle** and an **equilateral triangle** with minimized combined area.

5. Use the wire to form a **square** and an **equilateral triangle** with a minimized combined area.

6. Use the wire to form a **circle** and another **regular polygon** with minimized combined area.<